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Pseudo-Regular Oscillations Induced by External Noise

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An examination of the effect of noise on a general system at a saddle-node bifurcation has revealed that, in the limit of weak noise, the probability density of the time to pass through the saddle-node has a universal shape, the specific kinetics of the particular system serving only to set the time scale. This probability density is displayed and its salient features are explicated. In the case of a saddle-node bifurcation leading to relaxation oscillations, this analysis leads to the prediction of the existence of noise-induced oscillations which appear much less random than might at first be expected. The period of these oscillations has a well-defined, nonzero most probable value, the inverse of which is a noise-induced frequency. This frequency can be detected as a peak in power spectra from numerical simulations of such a system. This is the first case of the prediction and detection of a noise-induced frequency of which the authors are aware.

KEY WORDS: Noise; nonlinear oscillations; stochastic processes; bifurcation theory.

1. INTRODUCTION

A number of authors have suggested that noise-induced oscillations may appear when noise acts on a system at an unstable fixed point.[1-3] Such oscillations have been observed in numerical simulations of a simple model of fluid flow.[4]

In this communication, we present the results of a theoretical analysis of the appearance of noise-induced oscillations in systems at a particular bifurcation, the saddle-node bifurcation leading to relaxation oscillations. Our analysis makes it possible to predict the statistics of the noise-induced

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period when the noise is weak. An understanding of these statistics, in turn, makes it possible to understand why the oscillations appear considerably less random than one might at first expect.

This work is an application of an analysis that we have made of the effect of noise on systems at a general saddle-node bifurcation.[5,6] In Section 2, we present the relevant results of this analysis. In Section 3, we show how these results apply to the case of a saddle-node bifurcation leading to relaxation oscillations. This leads us to predict that a power spectrum taken from such a system should show a peak corresponding to a noise-induced frequency, the value of which can be predicted if the strength of the noise and the dynamics of the system at the saddle-node point are known. In Section 4, we confirm this prediction by presenting a power spectrum taken from a numerical simulation of such a system.

2. STATISTICS OF THE TIME TO PASS THROUGH A SADDLE NODE

For a system at a saddle-node bifurcation, the saddle-node point, like any fixed point in an essentially one-variable system, represents an impenetrable barrier. However, when such a system is subject to noise, passage through the saddle-node becomes possible. We have proven that, in the limit of weak noise, the shape of the probability density of the time to pass through the saddle node is independent of the specific functional form of the drift and diffusion and of the points between which the time is measured.[5,6] This result reflects the fact that, in the low-noise limit, the system spends almost all its time in a vanishingly small region around the saddle node. In such a region, the noisy dynamics is well approximated by the stochastic differential equation

$$\dot{x} = \tfrac{1}{2}kx^2 + \sigma\xi_t \tag{1}$$

where ξ_t is zero-mean, unit-variance, Gaussian white noise and the drift and diffusion have been approximated by the first nontrivial terms in their Taylor expansions around the saddle node ($x = 0$). For this model, the mean and variance of τ, the time to pass through the saddle node, can be calculated analytically; the results are

$$\begin{aligned} \langle \tau \rangle &= 6\left(\frac{1}{81}\right)^{1/3}\left[\Gamma\left(\frac{1}{3}\right)\right]^2\left(\frac{1}{\sigma^2 k^2}\right)^{1/3} \\ &\simeq 9.9521079\left(\frac{1}{\sigma^2 k^2}\right)^{1/3} \end{aligned} \tag{2}$$

and

$$\operatorname{var}(\tau) = 12\left(\frac{1}{81}\right)^{2/3}\left[\Gamma\left(\frac{1}{3}\right)\right]^4\left(\frac{1}{\sigma^2 k^2}\right)^{2/3}$$

$$= \frac{1}{3}\langle\tau\rangle^2 \tag{3}$$

Here, the time scale $(\sigma^2 k^2)^{-1/3}$ is the only quantity that depends on the specifics of the system. These predictions have been confirmed by numerical simulations.

The full probability density of τ may be obtained by numerical simulation of a system subject to noise at a saddle-node bifurcation. The resulting histogram, representing 100,000 samples of τ, is shown in Fig. 1. In this particular case, the mean of τ was 214.4117 sec. The probability shows two important features:

1. The density has a single, well-defined maximum at a nonzero time. This may be measured graphically to lie at $t = 123.3 \pm 3.3$ sec or at

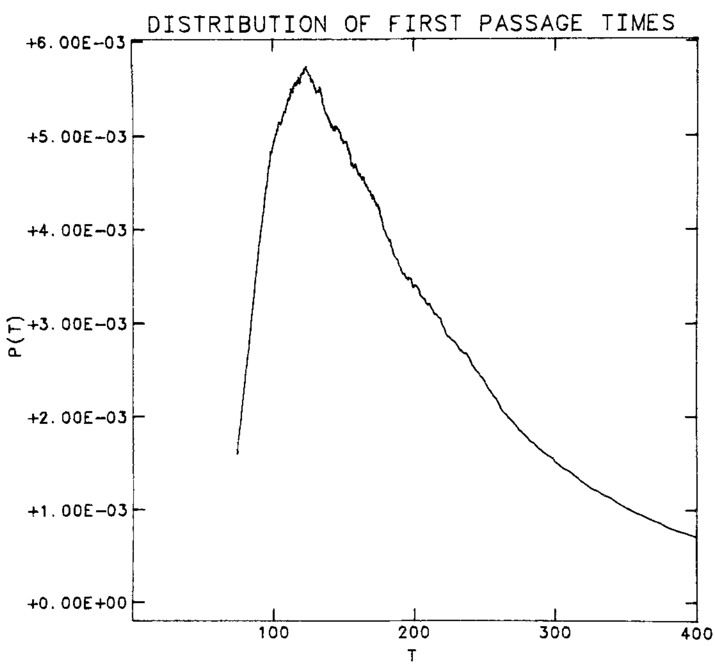

Fig. 1. Probability density for the time to pass through a saddle-node. The mean time is 214.4 sec.

0.5751 ± 0.015 times the mean. Since the shape of the density is independent of the specific kinetics of the system, this ratio will be the same for any system subject to weak noise at a saddle-node bifurcation.

2. There is an effective cutoff time before which passage through the saddle node is highly improbable. A theoretical analysis indicates that this cutoff occurs at 0.3058 times the mean time. In the case at hand, this implies a lower cutoff at $t = 65.5$ sec.

3. APPLICATION TO THE SADDLE-NODE BIFURCATION LEADING TO RELAXATION OSCILLATIONS

If we apply noise to a system at a saddle-node bifurcation leading to relaxation oscillations, we expect to observe noisy oscillations that closely resemble the relaxation oscillations that appear when the system is above the bifurcation.

We have simulated the effect of noise on a system at a saddle-node bifurcation leading to relaxation oscillations by numerically integrating the stochastic differential equation

$$\dot{\theta} = 1 - \cos\theta + \sigma\xi_t \tag{4}$$

and by examining the resulting time series for $x = \cos\theta$. Equation (4), a special case of the Adler equation, has a saddle-node point at $\theta = 0$. It provides a good model for the dynamics on the center manifold—the manifold that becomes the limit cycle above the bifurcation—in a system at

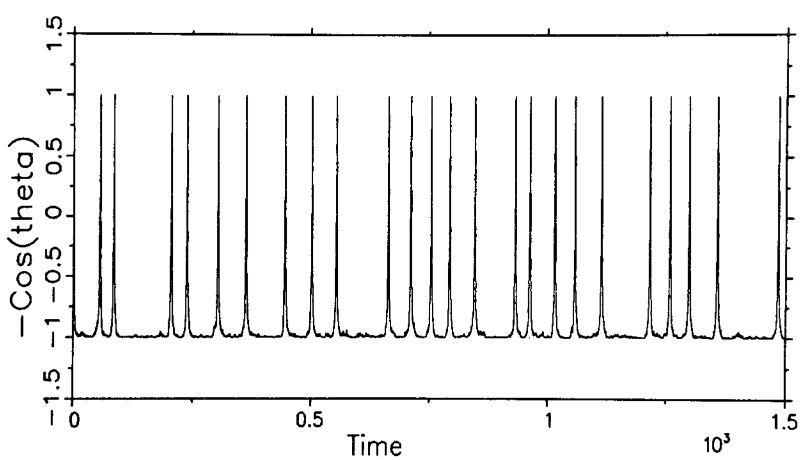

Fig. 2. Typical time series for $x = \cos\theta$ in Eq. (4). Note that the actual variable plotted is $-x$.

a saddle-node bifurcation leading to relaxation oscillations. For sufficiently weak noise, the system is confined to the center manifold, so (4) effectively describes the dynamics of the whole system.

A typical time series for $-x$ is shown in Fig. 2. Notice that the peaks do not appear to have been laid down on the time axis completely at random. If the peaks *were* distributed on the time axis at random, they would constitute a Poisson process; the probability density of the time between successive peaks would then be a decaying exponential and the most probable time would be $t=0$. In this case, however, the times between peaks seem to be clustered around some definite time scale. Such a time scale may be reasonably considered to be a noise-induced period.

It is important to understand that this is occurring in a system in which *there is no deterministic time scale*. Since the relaxation oscillations that arise deterministically appear at zero frequency (infinite period), any time scale must involve the noise as an essential element.

The pseudo-regular character of the noise-induced oscillations and, in particular, the existence of a nonzero most probable time between peaks follow directly from the results presented in the previous section. The time between peaks is essentially the time to pass through the saddle-node, so the statistical properties of the time to pass through a saddle-node enumerated in the previous section must apply to the time between successive peaks. In particular, there must be a nonzero most probable time between peaks equal to 0.5751 times the mean time, the mean time being given by Eq. (2). This most probable time will then be proportional to the noise-determined time scale $(\sigma^2 k^2)^{-1/3}$. Moreover, Eq. (3) indicates that the relative variance in the time between peaks must be one-third. If the peaks were laid down on the time axis completely at random, the relative variance in the time between peaks would be one. Thus, it is clear that, in this situation, the oscillations produced by noise are substantially less random than one might naively expect.

4. A NOISE-INDUCED FREQUENCY

The inverse of a noise-induced period should be a noise-induced frequency. The simulations used to produce the time series in Fig. 2 were run under conditions where the mean time between peaks is predicted, from Eq. (2), to be 73.33 in dimensionless time units. The most probable time should then be approximately 42.17 and we expect to find a noise-induced frequency at about 2.371×10^{-2} in dimensionless frequency units.

Figure 3 shows a power spectrum from a time series produced under the conditions described above. A graphical estimation places the location

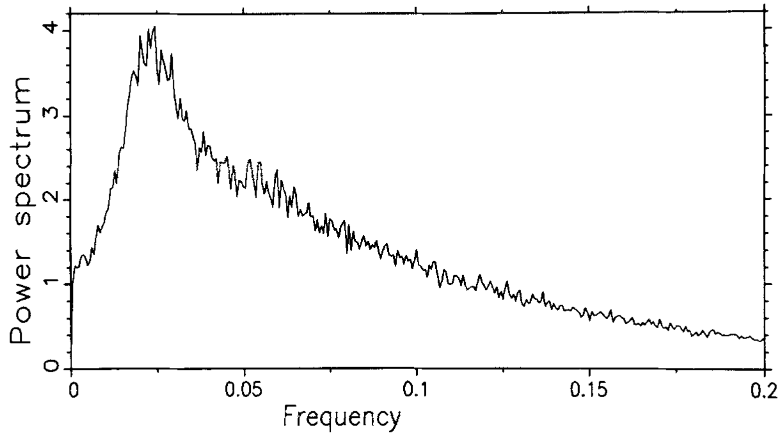

Fig. 3. Power spectrum calculated from a time series for $x = \cos\theta$ in Eq. (4). The frequency units are dimensionless.

of the peak at a frequency of $2.38 \times 10^{-2} \pm 0.6 \times 10^{-2}$, in excellent agreement with the value predicted above.

This is the first time, to the best of the authors' knowledge, that a noise-induced frequency, detectable as a peak in a power spectrum, has been described in the literature.

REFERENCES

1. F. H. Busse and K. E. Heikes, *Science* **208**:173 (1980).
2. F. H. Busse, in *Chaos and Order in Nature*, H. Haken, ed. (Springer-Verlag, Berlin, 1981).
3. G. Mayer-Kress and H. Haken, *Phys. Lett.* **82A**:151 (1981).
4. E. Stone and P. Holmes, submitted to *SIAM J. of App. Math.*
5. D. Sigeti, Dissertation, University of Texas at Austin (1988).
6. D. Sigeti and W. Horsthemke, to be published.